

An Interval Arithmetic Spreadsheet for Forecasting with Regional Input–Output Models

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ABSTRACT

Input-Output analysis is an important tool for regional economic development policy makers. Reasonable policy recommendations should include an idea of how realistic the results of the analysis are. These results will be affected by uncertainty in the technical coefficient matrix, changes in the technical coefficient matrix, uncertainty in final demand, and changes in final demand. This research demonstrates how a new spreadsheet program using interval arithmetic can help address these concerns.

INTRODUCTION

Let x_i represent the total supply of the i th good in an economy. Let a_{ij} be the amount of the i th good needed to produce a unit of the j th good and x_j be the total output of the j th good. Let b_i be the final (consumption) demand for the i th good. Assuming equilibrium (supply = demand) for the i th good gives

$$x_i = \sum_{j=1}^n a_{ij}x_j + b_i$$

where there are n goods in the economy. This can be written in matrix form for all goods as

$$(I - A)x = b$$

where $(I - A)$ is the Leontief matrix.

Given values for final demand, b_i , and the technical coefficients (the a_{ij}), a solution for the inputs is sought. If these given values are precisely known then the mathematics for the solution are well known.

In national input-output models, the technical coefficients are not precisely known but are inferred from sample data and might be better described as probability distributions.

Final demand is subject to uncertainty. West [8] gives a solution in terms of probability distributions when the probability distributions of the technical coefficients and final demand are known.

In practice, a great deal of data would be required to estimate the moments of the elements of the Leontief matrix. This may be beyond the resources of regional analysts. Regional models are often inferred from the national tables or from other non-sample information. In this case, the probability distributions are not known and the level of uncertainty is increased. Interval analysis offer a method of considering uncertainty in such models.

INTERVAL ARITHMETIC

Interval arithmetic was developed by R. E. Moore while studying the propagation and control of truncation and roundoff error using floating point arithmetic [4] on a digital computer. Moore was able to generalize this work into an arithmetic independent of machine considerations [3].

One of the nice features of interval arithmetic is that the rules are easy to implement. In what follows upper case letters, such as X , will be used to indicate intervals while lower case letters, such as x will indicate point values. An interval is defined to be the real compact interval $X = [\underline{x}, \bar{x}]$ where \underline{x} and \bar{x} indicate the lower and upper endpoints. A *box* is a vector of intervals such as $X = (X_1, X_2)$. If X and Y are intervals, the binary operations for interval arithmetic are

$$\begin{aligned} X + Y &= [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \\ X - Y &= [\underline{x} - \bar{y}, \bar{x} - \underline{y}], \\ X \cdot Y &= [\min(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}), \\ &\quad \max(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y})], \\ X/Y &= [\underline{x}, \bar{x}] \cdot [1/\bar{y}, 1/\underline{y}] \text{ if } 0 \notin Y. \end{aligned}$$

If $0 \in Y$ then the result for division is either two inter-

vals $[-\infty, a]$ and $[b, +\infty]$ or a single interval $[-\infty, +\infty]$. The arithmetic resulting from division by zero is called *extended interval arithmetic*. The rules for this extended arithmetic are not discussed here. Details can be found in Ratschek and Rokne [6]. Interval expressions can be developed for functions of intervals but these are not needed here.

SOLUTIONS OF INTERVAL EQUATIONS

A solution to a set of equations obtained using the rules of interval arithmetic is called a *solution set*. For example, consider the interval equations

$$\begin{aligned} [2, 4]X_1 + [-1, 1]X_2 &= [-3, 3] \\ [-1, 1]X_1 + [2, 4]X_2 &= [0, 0]. \end{aligned} \quad (1)$$

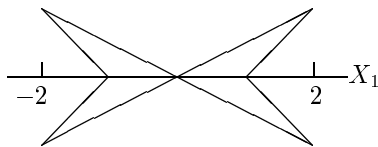


Figure 1: The solution set of system 1.

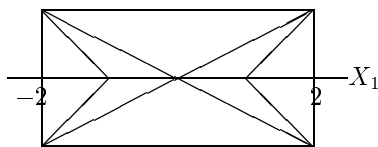


Figure 2: The hull solution of system 1.

The solution set is the set of all points that satisfy the interval equations. The solution set for the system of equations 1 in shown in Fig. 1. An *interval solution*, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, is a box found using the rules of interval arithmetic in some solution technique such as Gaussian elimination or Cramer's rule. An unfortunate feature of interval arithmetic is that different solution techniques will yield different boxes. All of the boxes, however, will enclose the solution set if the interval matrix \mathbf{A} is *regular*, that is if every point matrix A belonging to \mathbf{A} has rank n [5]. The smallest interval solution which contains the solution set is call the *hull solution*; the hull solution for the system of equations 1 is shown in Fig. 2. Barth and Nuding [7] have shown the conditions for which the hull solution exists. Jerrell [1][2] has shown that the input-output model meets these conditions.

AN INTERVAL SPREADSHEET

While the hull solution is desirable, at this time it can only be obtained using a interval procedures in a programming language such as Fortran or C. Many potential users of interval arithmetic may not wish to engage in the programming necessary to code these procedures or to use them. Interval Solver is an add-in spreadsheet program for Microsoft Excel under Windows NT/95 available from Delisoft Ltd, Helsinki, Finland. Interval Solver uses intervals as built in data types and performs computations using interval arithmetic. Interval Solver's computed results will most likely be larger than the hull solution (the hull solution cannot be easily computed for a general linear system of equations).

.0987	.0000	.0004	.0036	.0000	.0010	.0006	.0002	.0001
.0000	.0053	.0003	.0001	.0013	.0000	.0000	.0000	.0001
.0039	.0013	.0006	.0018	.0079	.0015	.0260	.0046	.0147
.0089	.0022	.0440	.1018	.0141	.0107	.0018	.0108	.0013
.0220	.0119	.0355	.0444	.0721	.0315	.0074	.0220	.0247
.0075	.0022	.0537	.0115	.0040	.0047	.0003	.0067	.0005
.0569	.0150	.0176	.0129	.0243	.0315	.0451	.0379	.0018
.0151	.0052	.0790	.0185	.0271	.0234	.0111	.0537	.0017
.0062	.0026	.0028	.0051	.0048	.0064	.0037	.0078	.0037

Figure 3: The technical coefficient matrix (A) for Coconino County.

X_1	[0.998659, 1.22058]
X_2	[0.69E - 005, 2.07E - 005]
X_3	[0.00591642, 0.00723126]
X_4	[0.0110387, 0.0134918]
X_5	[0.0258051, 0.0315395]
X_6	[0.0082211, 0.010048]
X_7	[0.0614338, 0.0750858]
X_8	[0.0183205, 0.0223917]
X_9	[0.00683657, 0.00835581]

Table 1: Change in output by industrial sector

X_1	[0.997567, 1.22192]
X_2	[1.69E - 005, 2.07E - 005]
X_3	[0.00590994, 0.00723919]
X_4	[0.0110267, 0.0135066]
X_5	[0.0257768, 0.0315741]
X_6	[0.0082121, 0.010059]
X_7	[0.0613666, 0.0751681]
X_8	[0.0183005, 0.0224163]
X_9	[0.0068291, 0.00836497]

Table 2: Change in output by industrial sector

A Matrix	Output Multiplier Industry 1
$\pm 1\%$	[[1.25937, 1.26564]]
$\pm 2\%$	[[1.25625, 1.26880]]
$\pm 10\%$	[[1.23117, 1.29450]]

Table 3: Effects of uncertainty in A matrix

APPLICATION OF THE SPREADSHEET

An aggregated technical coefficient (A) matrix for Coconino County, Arizona is shown in Fig. 3. The first example will consider the change in output in the county caused by a change in final demand in the first industrial sector, b_1 , holding the change in final demand constant for the remaining sectors. Regional analysts may not know final demand but can ask the question “what if final demand changes by a certain amount.” In this case suppose that the change is $\Delta b_1 = [.9, 1.1]$ units. The change in output by industrial sector is shown in Table 1.

Interval Solver can also compute results for interval values for elements of the technical coefficient matrix. This is useful when the user can attach some uncertainty the elements. Suppose that it was felt that element a_{11} is only known within $\pm 1\%$ of the value shown in the technical coefficient matrix for Coconino County; that is $a_{11} = [0.097713, 0.099687]$. Again let $\Delta b_1 = [.9, .11]$. The results for the change in output for the first industrial sector is given in Table 2.

As a final example, suppose that all of the elements of A are subject to uncertainty. Table 3 shows the effects of this uncertainty on the output multiplier for Industry 1. In this case every element of A has been given an interval value of a_{ij} plus or minus the value indicated in the left column of Table 3. It is important to understand that interval arithmetic guarantees *mathematically* that the true values of these variable will be inclosed in the interval results.

References

[1] M. E. Jerrell. Applications of interval computations to regional economic input-output models. In R. Baker Kearfott and Vladik Kreinovich, editors, *Applications of Interval Computations*, volume 3 of *Applied Optimization*, pages 133–142. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[2] M. E. Jerrell. Interval arithmetic for input-output models with inexact data. *Computational Economics*, 10:89–100, 1997.

[3] R. E. Moore. *Methods and Applications of Interval Analysis*. SIAM, Philadelphia, 1979.

[4] Ramon E. Moore. Automatic error analysis in digital computation. Technical Report LMSD-4882, Lockheed Missiles and Space Division Sunnyvale, California, 1995.

[5] Arnold Neumaier. *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, 1990.

[6] H. Ratschek and J. Rokne. *New Computer Methods for Global Optimization*. John Wiley & Sons, Inc., New York, 1988.

[7] Barth W. and Nuding E. Optimale lösung von intervallgleichungssystemen. *Computing*, 24:117–125, 1984.

[8] Guy R. West. A stochastic analysis of an input-output model. *Econometrica*, 54(2):363–374, 1986.